Name______________________________ Class__________ Date______________

AP Calculus ~ Semester I Review
Part 2
Implicit Derivatives and Related Rates

Product

1. Find the slope of a line tangent to the curve $x^2y + xy = 8$ at the point (1, 4).

$$\left[x^2\left(\frac{dy}{dx}\right) + y(2x)\right] + \left[x\frac{dy}{dx} + y(1)\right] = 0$$

$$x^2\frac{dy}{dx} + x\frac{dy}{dx} = -2xy - y$$

$$\frac{dy}{dx}(x^2+x) = -2xy - y$$

$$\frac{dy}{dx} = \frac{-2xy-y}{x^2+x}$$

$$\frac{dy}{dx} = \frac{-2(1)(4)-(4)}{(1)^2+(1)}$$

$$= \frac{-8-4}{2} = \frac{-12}{2} = \boxed{-6}$$

2. Find $\frac{d^2y}{dx^2}$ for $x^2 + xy = 5$

$$2x + x\frac{dy}{dx} + y(1) = 0$$

$$\frac{dy}{dx} = \frac{-y-2x}{x} \text{ or } \frac{-y}{x} - 2$$

$$\frac{d^2y}{dx^2} = \frac{x(-1)\frac{dy}{dx} - (-y)(1)}{x^2} - 0 \Rightarrow \frac{-x\frac{dy}{dx} + y}{x^2} \Rightarrow \frac{-x\left(\frac{-y}{x}-2\right)+y}{x^2} = \boxed{\frac{y+2x+y}{x^2}}$$

Related Rates

(A calculator is allowed on problems 3-7.)

3. A spherical balloon is being inflated at a rate of 20 cubic feet per minute.

a) In terms of circumference what is the rate at which the radius is changing in $\frac{ft}{min}$?

$$\frac{dV}{dt} = +20 \text{ ft}^3/\text{min}$$

$$C = 2\pi r$$

$$\frac{C}{2\pi} = r$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

$$20 = \frac{4}{\cancel{3}}\pi(\cancel{3}r^2)\frac{dr}{dt}$$

$$\frac{5}{r^2} = \frac{dr}{dt}$$

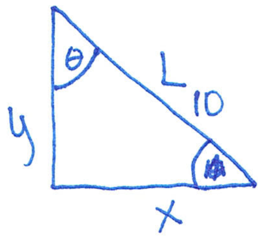
$$\frac{5}{\left(\frac{C}{2\pi}\right)^2} = \frac{dr}{dt}$$

$$\boxed{\frac{20\pi^2}{C^2} = \frac{dr}{dt}}$$

b) How fast is the radius increasing at the instant the radius is 2 ft?

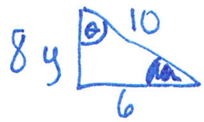
$$\boxed{\frac{dr}{dt} = \frac{5}{2^2} = \frac{5}{4} \text{ ft/min}}$$

4. A 10 foot ladder is leaning against a vertical wall. If the foot of the ladder slides away from the base of the wall at a rate of 1 $^{ft}/_{s}$, how fast is the top of the ladder sliding down the wall when the ladder is 6ft from the wall?



$\frac{dx}{dt} = 1\ ft/s \quad x = 6ft$

$\frac{dy}{dt} = ?$



$x^2 + y^2 = 10^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2(6)(1) + 2(8)\frac{dy}{dt} = 0$

$y^2 + 36 = 100$

$y^2 = 64 \quad y = 8$

$\boxed{\frac{dy}{dt} = \frac{-3}{4}\ ft/s}$

At what rate is the angle between the top of the ladder and the wall changing at this time?

$\sin\theta = \frac{x}{10}$

$\cos\theta \frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{1}{10}(1)\frac{-1}{\cos\theta}$

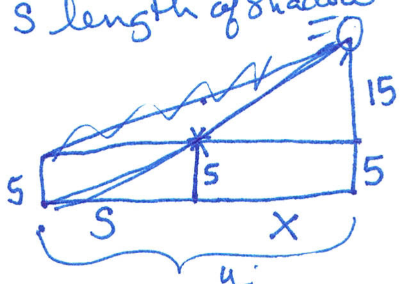
$\frac{d\theta}{dt} = \frac{1}{10}(1)\frac{-1}{\frac{8}{10}}$

$\frac{d\theta}{dt} = \frac{1}{10}(1)\frac{-10}{8} = \boxed{\frac{-1}{8}\ rad/sec}$

5. A forklift with a height of 5 ft is moving toward a light that is 20 foot off the floor, at a rate of 2 feet per second. At what rate is the shadow of the forklift shrinking at the moment the forklift is 15 feet from the point on the floor directly under the light?

* s+x tip of shadow

s length of shadow



$\frac{dx}{dt} = 2\ ft/sec$

$\left.\frac{ds}{dt}\right|_{x=15}$

$\frac{5}{s} = \frac{20}{s+x}$

$5s + 5x = 20s$

$5x = 15s$

$5\frac{dx}{dt} = 15\frac{ds}{dt}$

$5(-2) = 15\frac{ds}{dt}$

$\frac{-2}{3}\ ft/s = \frac{ds}{dt}$

At what rate is the farthest tip of its shadow approaching the point on the floor directly under the light at this time?

$\frac{d(s+x)}{dt} = \frac{dy}{dt}$

$\frac{5}{s} = \frac{20}{y}$

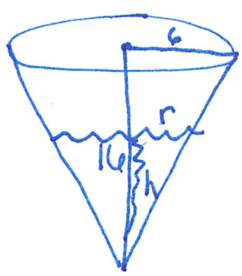
$5y = 20s$

$5\frac{dy}{dt} = 20\frac{ds}{dt}$

$5\frac{dy}{dt} = 20\left(\frac{-2}{3}\right)$

$\frac{dy}{dt} = \frac{-40}{3 \cdot 5} = \boxed{\frac{-8}{3}\ ft/sec}$

6. A minimum wage employee is filling a conical cup with pineapple flavored ice at a rate of 10 cubic centimeters per second. The conical cup has a diameter of 12 centimeters and a depth of 16 centimeters. At what rate is the depth of flavored ice in the container changing when the cup is 1/3 of its total volume?



$\frac{dV}{dt} = +10\ cm^3/sec$

r=6 h=16 cup

$\left.\frac{dh}{dt}\right|_{V=\frac{1}{3}total} =$

$V = \frac{1}{3}\pi(6)^2(16) = \pi\, 12 \cdot 16$

$\frac{1}{3}V = \frac{1}{3}\pi(12)(16) = 64\pi$

$\frac{6}{16} = \frac{r}{h}$ or $\frac{3}{8} = \frac{r}{h}$

$r = \frac{3}{8}h$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{3}{8}h\right)^2 h$

$V = \frac{1}{3}\pi\frac{9}{64}h^3 = \frac{3}{64}\pi h^3$

$\frac{dV}{dt} = \frac{9}{64}\pi h^2\frac{dh}{dt}$

$10 = \frac{9}{64}\pi(\ \ )^2\frac{dh}{dt}$

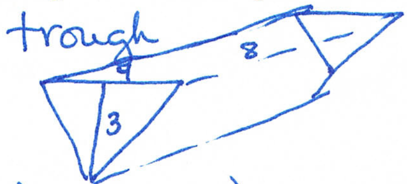
$64\pi = \frac{3}{64}\pi h^3$

$\frac{64^2}{3} = h^3$

$\sqrt[3]{\frac{(64)^2}{3}} = h$

7. A triangular prismatic horse trough is leaking at a rate of 3 cubic feet per hour. The triangular sides have a base of 4 ft. The trough has a total depth of 3 feet and an overall length of 8 feet. At what time is the trough $^2/_3$ full?

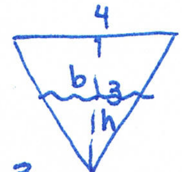
$\frac{dV}{dt} = -3\ ft^3/h$



$V = (\text{Area of base})(\text{height})$

$V = \frac{1}{2}(4)(3)(8) = 48\ ft^3$ full

$V(\frac{2}{3}\text{ full}) = \frac{2}{\cancel{3}} \cdot \cancel{48}\,16 = 32\ ft^3$



$\frac{4}{b} = \frac{3}{h}$ $\quad 3b = 4h \quad b = \frac{4h}{3}$

$32 = \frac{16}{3}h^2$

$6 = h^2 \quad h = \sqrt{6}$

What is the rate of change of the depth of water in the trough at this time?

$\frac{dh}{dt}\Big|_{V = 32ft^3}$

$V = \frac{1}{2}bh(8)$

$V = 4bh$

$V = 4 \cdot \frac{4}{3}h \cdot h$

$V = \frac{16}{3}h^2$

$\frac{dV}{dt} = \frac{32}{3}h\frac{dh}{dt}$

$-3 = \frac{32}{3}(\sqrt{6})\frac{dh}{dt}$

$\boxed{\frac{-9\ ft/hour}{32\sqrt{6}} = \frac{dh}{dt}}$

8. For $4y^3 + 12x^2y - 24x^2 + 12y = -1$,

a) find $^{dy}/_{dx}$.

$12y^2\frac{dy}{dx} + 12(x^2\frac{dy}{dx} + y\,2x) - 48x + 12\frac{dy}{dx} = 0$

$12y^2\frac{dy}{dx} + 12x^2\frac{dy}{dx} + 24xy - 48x + 12\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-24xy + 48x}{12y^2 + 12x^2 + 12}$

$\frac{-2xy + 4x}{y^2 + x^2 + 1} = \frac{dy}{dx}$

b) write the equation of each horizontal tangent.
(Show all computation and justify your answer.)

$-2xy + 4x = 0$

$2xy - 4x = 0$

$2x(y-2) = 0$

$x = 0 \quad y = 2$

$* x = 0 \Rightarrow 4y^3 + 0 - 0 + 12y = -1$

$4y^3 + 12y + 1 = 0$

$\boxed{GC \Rightarrow\ y = -.083}$

$y = 2 \quad 4(8) + \cancel{12x^2(2)} - \cancel{24x^2} + 12(2) = -1$

$\neq -1$

No Solution

Does not work

c) The line through the origin with slope of 1 is tangent to the curve at P. Find the x and y coordinates of P.
(Show all computation and justify your answer.)

$(0,0) \quad m = 1 \quad \ell \quad y - 0 = 1(x - 0)$

$y = x$

$4x^3 + 12x^3 - 24x^2 + 12x + 1 = 0$

$8x^3 - 24x^2 + 12x + 1 = 0$

$-2xy + 4x = y^2 + x^2 + 1$

$y = x \quad -2x^2 + 4x = x^2 + x^2 + 1$

$-4x^2 + 4x - 1 = 0$

$4x^2 - 4x + 1 = 0$

$(2x-1)(2x-1) = 0 \quad x = \frac{1}{2}$

$\boxed{x = \frac{1}{2} \quad y = \frac{1}{2}}$