Name Key

CALCULUS: LIMITS, CONTINUITY & DIFFERENTIABILITY PACKET

1. Explain in your own words what is meant by the equation

   $\lim_{x \to 2} f(x) = 5$. Limit as x approaches 2 is 5

   Is it possible for this statement to be true and yet $f(2) = 3$? Explain. Yes - There could be a hole at (2,5) and a point at (2,3)

2. Explain what it means to say that

   $\lim_{x \to 1^-} f(x) = 3$ and $\lim_{x \to 1^+} f(x) = 7$. Limit as x approaches 1 from the left is 3 and the limit as x approaches 1 from the right is 7.

   In this situation, it is possible that $\lim_{x \to 1} f(x)$ exists?

   No.- The limit as x approaches 1 must be the same from the left and right must be the same in order for the limit to exist

3. Explain the meaning of each of the following.

   (a) $\lim_{x \to -3} f(x) = \infty$ There is a VA at $x = -3$ and both sides are going up around the asymptote

   (b) $\lim_{x \to 4^+} f(x) = -\infty$ There is a VA at $x = 4$ and the right side of the graph is going down as it gets closer to the asymptote

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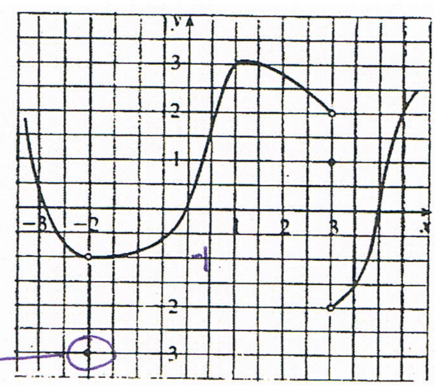
4. For the function f whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.



(a) $\lim_{x \to 1} f(x) = 3$ (b) $\lim_{x \to 3^-} f(x) = 2$ (c) $\lim_{x \to 3^+} f(x) = -2$

(d) $\lim_{x \to 3} f(x) = \text{DNE}$ (e) $f(3) = 1$ (f) $\lim_{x \to -2^-} f(x) = -1$

(g) $\lim_{x \to -2^+} f(x) = -1$ (h) $\lim_{x \to -2} f(x) = -1$ (i) $f(-2) = -3$

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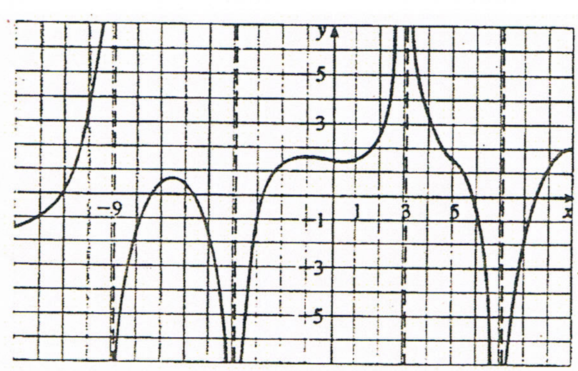
5. For the function f whose graph is shown, state the following.



(a) $\lim_{x \to 3} f(x) = +\infty$ (b) $\lim_{x \to 7} f(x) = -\infty$

(c) $\lim_{x \to -4} f(x) = -\infty$ (d) $\lim_{x \to -9^-} f(x) = +\infty$

(e) $\lim_{x \to -9^+} f(x) = -\infty$ $\lim_{x \to -9} f(x) = \text{DNE}$

(f) The equations of the vertical asymptotes

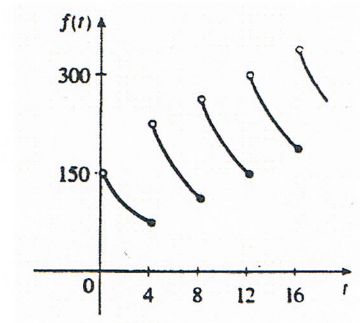
$x = -9, x = -4, x = 3, x = 7$

TURN--->>>

6. A patient receives a 150-mg injection of a drug every four hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. Find

$$\lim_{t\to 12^-} f(t) \text{ and } \lim_{x\to 12^+} f(t)$$

and explain the significance of these one-sided limits.



$\lim_{t\to 12^-} f(t) = 150$ $\lim_{t\to 12^+} f(t) = 300$

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7. Sketch the graph of the function $f(x) = \dfrac{1}{\left(1+2^{1/x}\right)}$ and state the value of each limit, if it exists. If it does not exist, explain why.

(a) $\lim_{x\to 0^-} f(x) = 1$ (b) $\lim_{x\to 0^+} f(x) = 0$ (c) $\lim_{x\to 0} f(x) =$ Does not exist

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8. Sketch the graph of the following function and use it to determine the values of a for which $\lim_{x\to a} f(x)$ exists.

$$f(x) = \begin{cases} 2-x, & x<-1 \\ x, & -1\le x<1 \\ (x-1)^2, & x\ge 1 \end{cases}$$

$2-x$ | x | $(x-1)^2$

-3 -1 -1 1 0

$\lim_{x\to -1} f(x) = DNE$ $\lim_{x\to 1} f(x) = DNE$

Fill in the table for the following functions to find the given limit.

9. $f(x) = \dfrac{\sin(3x)}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	2.955	2.9996	2.99999	Error	2.99999	2.9996	2.955

$\lim_{x\to 0} \dfrac{\sin(3x)}{x} = 3$

L'Hopitals

10. $g(x) = \dfrac{1-\cos x}{x^2}$ $\lim_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{0}{0}$ $\lim_{x\to 0} \dfrac{\sin x}{2x} = \dfrac{0}{0}$ $\lim_{x\to 0} \dfrac{\cos x}{2} = \dfrac{1}{2}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$g(x)$	.49958	.49999	.49999	Error	.49999	.49999	.49958

$\lim_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$

TURN--->>>

11. Given that $\lim_{x\to a} f(x) = -3$, $\lim_{x\to a} g(x) = 0$, $\lim_{x\to a} h(x) = 8$, find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x\to a}\left[f(x)+h(x)\right] = 5$

$-3 + 8$

(b) $\lim_{x\to a}\left[f(x)\right]^2 = (-3)^2 = 9$

(c) $\lim_{x\to a}\sqrt[3]{h(x)} = \sqrt[3]{\lim_{x\to a} h(x)} = \sqrt[3]{8} = 2$

(d) $\lim_{x\to a}\dfrac{1}{f(x)} = \dfrac{1}{-3} = \dfrac{-1}{3}$

(e) $\lim_{x\to a}\dfrac{f(x)}{h(x)} = \dfrac{-3}{8}$

(f) $\lim_{x\to a}\dfrac{g(x)}{f(x)} = \dfrac{0}{-3} = 0$

(g) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{-3}{0}$ DNE

(h) $\lim_{x\to a}\dfrac{2f(x)}{h(x)-f(x)} = \dfrac{2(-3)}{8+(+3)} = \dfrac{-6}{11}$

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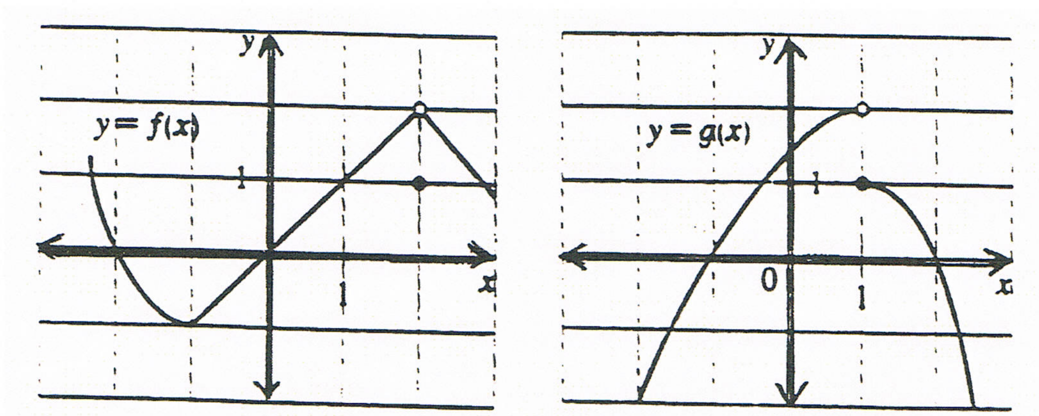
12. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.



(a) $\lim_{x\to 2}\left[f(x)+g(x)\right] = 2+0=2$

(b) $\lim_{x\to 1}\left[f(x)+g(x)\right] = 1+\text{DNE} = \text{DNE}$

(c) $\lim_{x\to 0}\left[f(x)g(x)\right] = 0(1.?)=0$

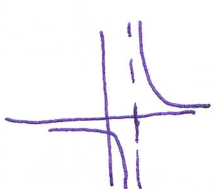
(d) $\lim_{x\to -1}\dfrac{f(x)}{g(x)} = \dfrac{-1}{0} = \text{DNE}$

(e) $\lim_{x\to 2} x^3 f(x) = 2^3(2) = 16$

(f) $\lim_{x\to 1}\sqrt{3+f(x)} = \sqrt{3+1} = 2$

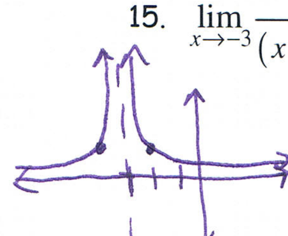
Find the limit. Draw a sketch for each problem. Do not use your calculator.

13. $\lim_{x\to 1^+}\dfrac{1}{x-1} = +\infty$



14. $\lim_{x\to 1}\dfrac{1}{x-1} = \text{DNE}$

15. $\lim_{x\to -3}\dfrac{1}{(x+3)^2} = +\infty$



TURN--->>>

$\frac{-1}{x-5}$

16. $\lim_{x\to 5^-} \frac{1}{5-x} = +\infty$

17. $\lim_{x\to 5^-} \frac{1}{(5-x)^2} = +\infty$

18. $\lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$

19. $\lim_{x\to 3} \frac{|x-3|}{x-3} = \text{DNE}$

20. $\lim_{x\to 2} [x+1] = \text{DNE}$

21. $\lim_{x\to 2^+} \frac{x^3|x-2|}{x-2} = 8$

$-x^3 \; x<2 \quad +x^3 \; x>2$

22. $\lim_{x\to 4^-} \frac{x^3[x-4]}{x-4} = +\infty$

23. $\lim_{x\to 3^+} \left(x-3-\frac{1}{x-3}\right) = -\infty$

24. $\lim_{x\to \frac{\pi}{2}^+} \tan x = -\infty$

25. $\lim_{x\to -\frac{\pi}{2}^+} \sec x = -\infty$

26. $\lim_{x\to \pi^-} \csc x = +\infty$

27. $\lim_{x\to 0^-} \cot x = -\infty$

28. $f(x) = \begin{cases} x^2-1 \text{ if } x<2 \\ 3x-2 \text{ if } x>2 \end{cases}$

$x^2-1 \mid 3x-2$

$3 \quad 2 \quad 4$

a) $\lim_{x\to 2^-} f(x) = 3$

b) $\lim_{x\to 2^+} f(x) = 4$

c) $\lim_{x\to 2} f(x) = \text{DNE}$

$x+3 \mid 3x^2+1$ Und

$4 \quad 1 \quad 4$

29. $g(x) = \begin{cases} x-3 \text{ if } x\neq 1 \\ 4 \text{ if } x=1 \end{cases}$ $\qquad \lim_{x\to 1} g(x) = -2$

$x-3 \mid x-3$ 4

$-2 \quad 1 \quad -2$

30. $h(x) = \begin{cases} x+3 \text{ if } x<1 \\ 3x^2+1 \text{ if } x>1 \end{cases}$ $\qquad \lim_{x\to 1} h(x) = 4$

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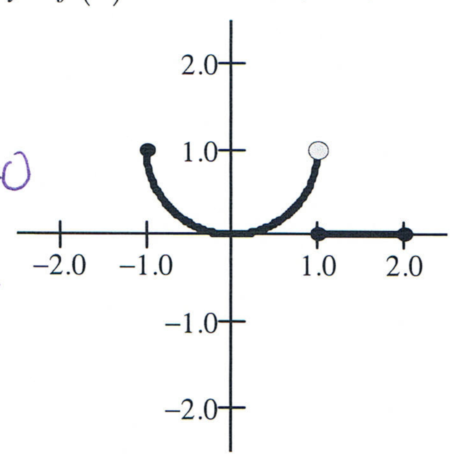
31. Determine if the following statements regarding the function $y = f(x)$ are true or false.



a. $\lim_{x\to -1^+} f(x) = 1$ True

b. $\lim_{x\to 0^-} f(x) = 0$ ~~False~~ True

c. $\lim_{x\to 0^-} f(x) = 1$ False $=0$

d. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$ True $=0$

e. $\lim_{x\to 0} f(x)$ exists Yes $=0$

f. $\lim_{x\to 0} f(x) = 0$ True

g. $\lim_{x\to 0} f(x) = 1$ False $=0$

h. $\lim_{x\to 1} f(x) = 1$ False DNE

i. $\lim_{x\to 1} f(x) = 0$ False $\lim_{x\to 1^+} f(x) = 0$

j. $\lim_{x\to 2} f(x) = ?$ $\lim_{x\to 2^-} f(x) = 0$

TURN--->>>

On problems 32 - 37 (Use your graphing calculator on problems 36 and 37):

(a) find $\lim_{x\to\infty} f(x)$

(b) find $\lim_{x\to-\infty} f(x)$

(c) identify all horizontal asymptotes.

$\lim_{x\to\pm\infty} f(x)=3$

32. $f(x)=\dfrac{3x^3-x+1}{x+3} = 3x^2$

$\lim_{x\to\infty} f(x)=+\infty$ $\lim_{x\to-\infty} f(x)=+\infty$

33. $f(x)=\dfrac{4x^2-3x+5}{2x^3+x-1} = \dfrac{1}{x}$

$\lim_{x\to\pm\infty} f(x)=0$

34. $f(x)=\dfrac{3x+1}{x-4} = \dfrac{3}{1}$

35. $f(x)=\dfrac{3x+1}{|x|+2} =$

$\lim_{x\to-\infty} f(x)=\dfrac{-3}{1}=-3$ $\lim_{x\to+\infty} f(x)=\dfrac{+3}{1}=+3$

36. $\cancel{f(x)=\dfrac{\sin 3x}{x}=3} = 0$

37. $f(x)=\cos\left(\dfrac{1}{x}\right)$

$\cos 0 = 1$

On problems 38 - 41 (Use your calculator on problems 37 - 39):

(a) find the vertical asymptotes of $f(x)$

(b) describe the behavior of $f(x)$ to the left and right of each vertical asymptote.

38. $f(x)=\dfrac{1}{x^2-4}$ $\lim_{x\to 2}$

$(x+2)(x-2)$

39. $f(x)=\dfrac{x^2+5x+6}{x^2-4}$ $\dfrac{(x+3)\cancel{(x+2)}}{(x-2)\cancel{(x+2)}}$

Hole at $x=-2$

UA at $x=2$

Opp directions

40. $f(x)=\dfrac{x^2-2x}{x+1}$ $x(x-2)$

$x=-1$ opp directi[on]

slant asymptote

41. $f(x)=\sec x$ $\cos x = 0$

$x=\frac{\pi}{2}, \frac{3\pi}{2}$, etc opp directions

42. Given $f(x)=\dfrac{2-5|x|}{3x+4}$, find the following. Show all work!

(a) $\lim_{x\to\infty} f(x) = \dfrac{-5}{3}$

(b) $\lim_{x\to-\infty} f(x) = \dfrac{-5}{-3} = \dfrac{5}{3}$

(c) Name the horizontal asymptotes of *f*. $y=-\frac{5}{3}$ $y=\frac{5}{3}$

(d) Name the vertical asymptotes of *f*. $x=-\frac{4}{3}$

On problems 43 and 44:

(a) Find the values of x, if any, for which the function is discontinuous.

(b) Identify each discontinuity as point, jump, or asymptotic.

(c) Identify each discontinuity as removable or nonremovable.

43. $f(x)=\dfrac{x^2-9}{x-3} = \dfrac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$

$= x+3$

$x=3$ removable discontinuity

point discontinuity

44. $g(x)=\dfrac{x-1}{x^2-x-2}$ $\dfrac{x-1}{(x-2)(x+1)}$

$x=2$

$x=-1$

Vertical asymptotes

Nonremovable

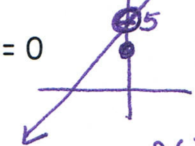
TURN--->>>

45) $f(0)$ exists $\lim_{x\to 0} f(x)$ exist $f(0) \neq \lim_{x\to 0} f(x)$

$4 \neq 5$

For problems 45 - 47, graph the function. Determine if the function is continuous at the specified value of x. Justify (making sure to show all three steps of the definition).

45. $g(x)=\begin{cases} x+5 & x\neq 0 \\ 4 & x=0 \end{cases}$ at x = 0



46. $h(x)=\begin{cases} -x^2+8 & x<2 \\ \frac{1}{2}x+3 & x\geq 2 \end{cases}$ at x = 2

$-4+8=4$

$\frac{1}{2}(2)+3=4$

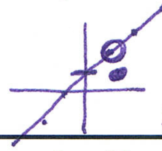
$f(2)=4$

$\lim_{x\to 2} f(2)=4$

$f(2) \stackrel{\checkmark}{=} \lim_{x\to 2} f(x)$

47. $f(x)=\begin{cases} \frac{x^2-1}{\cancel{x-1}} & x\neq 1 \\ x & x=1 \end{cases}$

$(\cancel{x-1})(x+1)$



$f(1)$ exists

$\lim_{x\to 1} f(x)$ exists

$f(1) \neq \lim_{x\to 1} f(x)$ $\{1 \neq 2\}$

$(\cancel{x-2})(x+5)$ $2+5$

48. $f(x)=\begin{cases} \frac{x^2+3x-10}{\cancel{x-2}} & x\neq 2 \\ ?\,7 & x=2 \end{cases}$ Define $f(2)$ in a way that intends $f(x)$ to be continuous.

$2+5=7$

It is cont at 3!

49. Find k so that f will be continuous at x = 3, given $f(x)=\begin{cases} kx^2, & x\leq 2 \\ 2x+k, & x>2 \end{cases}$.

~~$K(2)^2$ must be equal $2(2)+K$~~

~~$4K=4+K$ $3K=4$ $K=\frac{4}{3}$~~

$x+1$ | e^x | $2-x$

1 0 1 → e^1 1

50. Determine if $f(x)$ is continuous at $x=0$ and $x=1$. Justify. $f(x)=\begin{cases} x+1, & x<0 \\ e^x, & 0\leq x\leq 1 \\ 2-x, & x>1 \end{cases}$

e^1

$2-1$

$f(0)=\lim_{x\to 0} f(x)=1$

~~$f(1) \neq \lim$~~

$\lim_{x\to 1} f(x)$ DNE

$e^1 \neq 1$

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51. Find a value of constant k that will make the function continuous. $f(x)=\begin{cases} kx^2, & x\leq 2 \\ 2x+k, & x>2 \end{cases}$

$K(2)^2=2(2)+K$ $K=\frac{4}{3}$

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52. Find a value of constant k and m that will make the function continuous.

$f(x)=\begin{cases} x^2+5, & x>2 \\ m(x+1)+k, & -1<x\leq 2 \\ 2x^3+x+7, & x\leq -1 \end{cases}$

① $2^2+5=m(2+1)+K$

$4+5=3m+K$

$9=3m+4$

$\boxed{\frac{5}{3}=m}$

② $m(0)+K=2(-1)^3+(-1)+7$

$K=-2-1+7$

$\boxed{K=4}$

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53. Find a value of constant k that will make the function continuous. $f(x)=\begin{cases} \frac{\sin 3x}{x}, & x\neq 0 \\ k, & x=0 \end{cases}$

$\lim_{x\to 0} \frac{\sin 3x}{x}=3$ $K=3$ in order to

fill in the hole

TURN--->>>

54. Find a value of constant c and d that will make the function continuous.

$$m(x)=\begin{cases} -\sqrt{4-(x+3)^2}, & x\le -1 \\ cx+d, & -1<x<3 \\ \sqrt{x-3}+4, & x\ge 3 \end{cases}$$

① $-\sqrt{4-(-1+3)^2} = -c+d$ ② $3c+d=\sqrt{3-3}+4$

$0=-c+d$ $3c+d=4$

$-$ $3c+d=4$
$-c+d=0$ $\rightarrow$ $-4c=-4$ $\boxed{c=1}$

$-1+d=0$ $\boxed{d=1}$

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55. Show whether the Intermediate Value Theorem holds. If the theorem holds, find the value of c which the theorem guarantees; if the theorem does not hold give the reason. Also sketch the graph of f.

$$f(x)=(x-3)^2+2,\ [a,b]=[1,4],\ k=5$$

Continuous there has to be all values between 3+6 so 5 has to be there

$f(1)=4+2=6$

$f(4)=1+2=3$

$(x-3)^2+2=5$ $(x-3)^2=3$ $x-3=\pm\sqrt{3}$ $x=\pm\sqrt{3}+3$ $\boxed{3-\sqrt{3}}$

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56. (Calculator allowed.) The population y, of bacteria *Makeyoucoughus hurtyourthruatus* is modeled by the equation $y=50e^{.1013663t}$, where t is days and y is the number of colonies of bacteria. Use the Intermediate Value Theorem to verify that the bacteria will reach a population of 100 colonies on the time interval [4,7]. Then determine when the population will reach 100 colonies.

$f(4)=75$ $f(7)=101.655$ e^x graphs are continuous $f(6.838)\approx 100$

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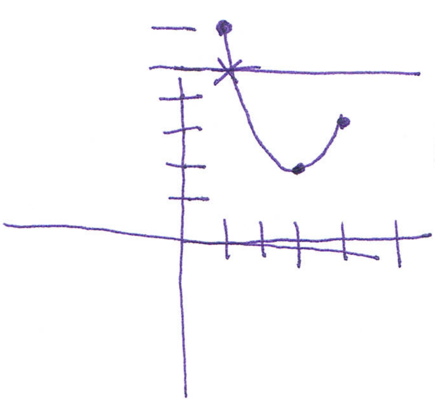
57. Just for fun….

James' Diabolical Challenge Problem:

Given $j(x)=\begin{cases} \dfrac{x^2-(4+A)x+4A}{x-4}, & x\neq 4 \\ B, & x=4 \end{cases}$ and $j(2)=1$ and $j(x)$ is everywhere continuous,

find A and B.

☺ You've Finished Your First Packet of Calculus! ☺



$\boxed{3-\sqrt{3}}$

$$j(4) = \frac{4^2 - (4+A)(4) + 4A}{4-4} = \frac{16 - 16 - 4A + 4A}{0} = \frac{0}{0}$$

$$j(2) = \frac{2^2 - (4+A)(2) + 4A}{2-4} = 1$$

$$4 - 8 - 2A + 4A = -2$$

$$-4 + 2A = -2$$

$$2A = 2$$

$$\boxed{A = 1}$$

$$(\cancel{x-4})(x-1)$$

$$j(x) = \begin{cases} \dfrac{x^2 - 5x + 4}{\cancel{x-4}} & x \neq 4 \\ B & x = 4 \end{cases}$$

$$\boxed{B = 3}$$