Sect. 12-4: Convergent and Divergent Series

If an infinite series has a sum, or limit, the series is CONVERGENT. If a series is not convergent, it is DIVERGENT.

Are arithmetic series convergent or divergent?

Are geometric series convergent or divergent?

<u>Examples</u>: Determine whether each series is convergent or divergent.

1) $-6 \overset{+3}{-} 3 \overset{+3}{-} 0 \overset{+3}{+} 3 + \dots$ arithmetic ∴ divergent

2) $2 \overset{\cdot 5}{+} 10 \overset{\cdot 5}{+} 50 + \dots$ geometric w/ $r = 5 > 1$ ∴ divergent

3) $\frac{1}{4} \overset{\cdot \frac{1}{3}}{+} \frac{1}{12} \overset{\cdot \frac{1}{3}}{+} \frac{1}{36} + \dots$ geometric w/ $r = \frac{1}{3} < 1$ ∴ convergent

4) $-8 \overset{+6}{-} 2 \overset{+6}{+} 4 + \dots$ arithmetic ∴ divergent

5) $1 \overset{\cdot 5}{+} 5 \overset{\cdot 5}{+} 25 + \dots$ geometric w/ $r = 5 > 1$ ∴ divergent

If a series is neither arithmetic nor geometric, we must have a different way to decide its convergence or divergence.

Ratio Test for Convergence of a Series

Let a_n and a_{n+1} represent two consecutive terms of a series of positive terms. Suppose $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and that $r = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$. Then the series is convergent if $r < 1$ & divergent if $r > 1$. If $r = 1$, the test provides no information.

Examples: Use the ratio test to determine whether each series is convergent or divergent.

$n=1 \quad n=2 \quad n=3 \quad n=4$

1) $\frac{2^1}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \ldots \frac{2^4}{4}$

$$r = \lim_{n\to\infty} \frac{2^{n+1}}{n+1} \cdot \frac{n}{2^n}$$

$a_n = \frac{2^n}{n}$

$$r = \lim_{n\to\infty} \frac{\cancel{2^n} \cdot 2^1}{n+1} \cdot \frac{n}{\cancel{2^n}}$$

$a_{n+1} = \frac{2^{n+1}}{n+1}$

$$r = \lim_{n\to\infty} \frac{\boxed{2}n^1}{\boxed{1}n^1+1} \qquad N=D$$

$r = 2 > 1 \therefore$ divergent

$n=1 \quad n=2 \quad n=3$

2) $\frac{1}{1^1}+\frac{1}{2^2}+\frac{1}{3^3}+\ldots\frac{1}{4^4}$

$$r=\lim_{n\to\infty}\frac{1}{(n+1)^{n+1}}\cdot\frac{n^n}{1}$$

$$a_n=\frac{1}{n^n}$$

$$r=\lim_{n\to\infty}\frac{n^n}{(n+1)^{n+1}} \qquad N<D$$

$$a_{n+1}=\frac{1}{(n+1)^{n+1}}$$

$r=0<1$ ∴ <u>convergent</u>

$n=1 \quad n=2 \quad n=3 \quad n=4$

3) $\frac{2}{1}+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\ldots\frac{6}{5}$

$$r=\lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{n}{n+1}$$

$$a_n=\frac{n+1}{n}$$

$$r=\lim_{n\to\infty}\frac{n^2+2n}{n^2+2n+1} \qquad N=D$$

$$a_{n+1}=\frac{(n+1)+1}{n+1}=\frac{n+2}{n+1}$$

$r=1$ <u>cannot be determined by the ratio test</u>

Remember: "n!" is read "n factorial" and means

$$n! = (n)(n-1)(n-2)\ldots(1)$$

1, 3, 5, 7 arithmetic

$a_n = 1 + (n-1)2$

$a_n = 1 + 2n - 2$

$a_n = 2n - 1$

$n=1 \quad n=2 \quad n=3 \quad n=4$

4) $1+\frac{2}{1\cdot2\cdot3}+\frac{3}{1\cdot2\cdot3\cdot4\cdot5}+\frac{4}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}+\ldots$

$1! \quad 3! \quad 5! \quad 7!$

$a_n = \frac{n}{(2n-1)!}$

$a_{n+1} = \frac{n+1}{(2(n+1)-1)!} = \frac{n+1}{(2n+1)!}$

$r = \lim_{n\to\infty} \frac{n+1}{(2n+1)!} \cdot \frac{(2n-1)!}{n}$

$r = \lim_{n\to\infty} \frac{(n+1)\,\cancel{(2n-1)(2n-2)(2n-3)\cdots1}}{(2n+1)(2n)\,\cancel{(2n-1)\cdots1}\; n}$

$r = \lim_{n\to\infty} \frac{n+1}{\underbrace{(2n+1)(2n)n}_{2n^2}} = \lim_{n\to\infty} \frac{n+1}{4n^3+2n^2}$

$N < D \quad r = 0 < 1 \;\therefore$ convergent

$r = \lim_{n\to\infty} \frac{\cancel{10^n}\cdot 10^1}{(n+1)\cancel{(n)(n-1)\cdots1}} \cdot \frac{\cancel{n(n-1)(n-2)\cdots1}}{\cancel{10^n}}$

$r = \lim_{n\to\infty} \frac{10n}{n+1} \quad N = D \quad r = 10 > 1$

$\therefore$ divergent

$r = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1}$

$r = \lim_{n\to\infty} \frac{\cancel{n(n-1)(n-2)\cdots1}}{(n+1)\cancel{(n)(n-1)\cdots1}}$

$r = \lim_{n\to\infty} \frac{1}{n+1} \quad N < D$

$r = 0 < 1 \;\therefore$ convergent

You try:

1) $\frac{10^1}{1!}+\frac{10^2}{1\cdot2}+\frac{10^3}{1\cdot2\cdot3}+\ldots$

$2! \quad 3!$

$a_n = \frac{10^n}{n!}$

$a_{n+1} = \frac{10^{n+1}}{(n+1)!}$

2) $\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots$

$a_n = \frac{1}{n!}$

$a_{n+1} = \frac{1}{(n+1)!}$