typed by Sean Bird, Covenant Christian High School
updated August 15, 2009

# AP CALCULUS
Stuff you **MUST** know Cold

* means topic only on BC

## Curve sketching and analysis

$y = f(x)$ must be continuous at each:

critical point: $\frac{dy}{dx} = 0$ or undefined

and look out for endpoints

local minimum:

$\frac{dy}{dx}$ goes (–,0,+) or (–,und,+) or $\frac{d^2y}{dx^2} > 0$

local maximum:

$\frac{dy}{dx}$ goes (+,0,–) or (+,und,–) or $\frac{d^2y}{dx^2} < 0$

point of inflection: concavity changes

$\frac{d^2y}{dx^2}$ goes from (+,0,–), (–,0,+), (+,und,–), or (–,und,+)

## Basic Derivatives

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\frac{d}{dx}\left(\sin x\right) = \cos x$$

$$\frac{d}{dx}\left(\cos x\right) = -\sin x$$

$$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$$

$$\frac{d}{dx}\left(\cot x\right) = -\csc^2 x$$

$$\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$$

$$\frac{d}{dx}\left(\csc x\right) = -\csc x \cot x$$

$$\frac{d}{dx}\left(\ln u\right) = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}\left(e^u\right) = e^u\frac{du}{dx}$$

where $u$ is a function of $x$, and $a$ is a constant.

## More Derivatives

$$\frac{d}{dx}\left(\sin^{-1}\frac{u}{a}\right) = \frac{1}{\sqrt{a^2-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\cos^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1}\frac{u}{a}\right) = \frac{a}{a^2+u^2}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left(\cot^{-1}x\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1}\frac{u}{a}\right) = \frac{a}{|u|\sqrt{u^2-a^2}}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left(\csc^{-1}x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(a^{u(x)}\right) = a^{u(x)}\ln a\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\ln a}$$

## Differentiation Rules

Chain Rule

$$\frac{d}{dx}\left[f(u)\right] = f'(u)\frac{du}{dx} \quad OR \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Product Rule

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx} \quad OR \quad u'v + uv'$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} \quad OR \quad \frac{u'v - uv'}{v^2}$$

## "PLUS A CONSTANT"

## The Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F'(x) = f(x)$

## Corollary to FTC

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f(b(x))b'(x) - f(a(x))a'(x)$$

## Intermediate Value Theorem

If the function $f(x)$ is continuous on $[a, b]$, and $y$ is a number between $f(a)$ and $f(b)$, then there exists at least one number $x = c$ in the open interval $(a, b)$ such that

$$f(c) = y.$$

## Mean Value Theorem

If the function $f(x)$ is continuous on $[a, b]$, AND the first derivative exists on the interval $(a, b)$, then there is at least one number $x = c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

## Rolle's Theorem

If the function $f(x)$ is continuous on $[a, b]$, AND the first derivative exists on the interval $(a, b)$, AND $f(a) = f(b)$, then there is at least one number $x = c$ in $(a, b)$ such that

$$f'(c) = 0.$$

## Approx. Methods for Integration

Trapezoidal Rule

$$\int_a^b f(x)dx = \tfrac{1}{2}\tfrac{b-a}{n}[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)]$$

Simpson's Rule

$$\int_a^b f(x)dx = \tfrac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

## Theorem of the Mean Value i.e. AVERAGE VALUE

If the function $f(x)$ is continuous on $[a, b]$ and the first derivative exists on the interval $(a, b)$, then there exists a number $x = c$ on $(a, b)$ such that

$$f(c) = \frac{\int_a^b f(x)dx}{(b-a)}$$

This value $f(c)$ is the "average value" of the function on the interval $[a, b]$.

## Solids of Revolution and friends

Disk Method

$$V = \pi\int_{x=a}^{x=b}\left[R(x)\right]^2 dx$$

Washer Method

$$V = \pi\int_a^b\left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right)dx$$

General volume equation (not rotated)

$$V = \int_a^b Area(x)\,dx$$

*Arc Length $L = \int_a^b\sqrt{1+\left[f'(x)\right]^2}\,dx = \int_a^b\sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\,dt$

## Distance, Velocity, and Acceleration

velocity = $\frac{d}{dt}$(position)

acceleration = $\frac{d}{dt}$(velocity)

*velocity vector = $\left\langle\frac{dx}{dt}, \frac{dy}{dt}\right\rangle$

speed = $|v| = \sqrt{(x')^2 + (y')^2}$ *

displacement = $\int_{t_o}^{t_f} v\,dt$

distance = $\int_{\text{initial time}}^{\text{final time}} |v|\,dt$

$\int_{t_o}^{t_f}\sqrt{(x')^2 + (y')^2}\,dt$ *

average velocity = $\frac{\text{final position} - \text{initial position}}{\text{total time}}$ $= \frac{\Delta x}{\Delta t}$

# BC TOPICS and important TRIG identities and values

### l'Hôpital's Rule

If $\dfrac{f(a)}{g(b)} = \dfrac{0}{0}$ or $= \dfrac{\infty}{\infty}$,

then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$

### Slope of a Parametric equation

Given a $x(t)$ and a $y(t)$ the slope is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

### Values of Trigonometric Functions for Common Angles

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| $0°$ | 0 | 1 | 0 |
| $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ |
| $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 |
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| $\frac{\pi}{2}$ | 1 | 0 | "$\infty$" |
| $\pi$ | 0 | $-1$ | 0 |

*Know both the **inverse trig** and the **trig** values. E.g.* $\tan(\pi/4)=1$ & $\tan^{-1}(1)=\pi/4$

### Euler's Method

If given that $\frac{dy}{dx} = f(x, y)$ and that the solution passes through $(x_o, y_o)$,

$y(x_o) = y_o$

$\vdots$

$y(x_n) = y(x_{n-1}) + f(x_{n-1}, y_{n-1}) \cdot \Delta x$

In other words:

$$x_{\text{new}} = x_{\text{old}} + \Delta x$$

$$y_{\text{new}} = y_{\text{old}} + \left.\frac{dy}{dx}\right|_{(x_{\text{old}}, y_{\text{old}})} \cdot \Delta x$$

### Polar Curve

For a polar curve $r(\theta)$, the **AREA** inside a "leaf" is

$$\int_{\theta_1}^{\theta_2} \tfrac{1}{2}\left[r(\theta)\right]^2 d\theta$$

where $\theta_1$ and $\theta_2$ are the "first" two times that $r =$ 0.

The **SLOPE** of $r(\theta)$ at a given $\theta$ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}\left[r(\theta)\sin\theta\right]}{\frac{d}{d\theta}\left[r(\theta)\cos\theta\right]}$$

### Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

### Integral of Log

Use IBP and let $u = \ln x$ (Recall $u$=LIPET)

$$\int \ln x\,dx = x\ln x - x + C$$

### Ratio Test

The series $\sum\limits_{k=0}^{\infty} a_k$ converges if

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1$$

If the limit equal 1, you know nothing.

### Trig Identities

*Double Argument*

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

*Pythagorean*

$\sin^2 x + \cos^2 x = 1$

(others are easily derivable by dividing by $\sin^2 x$ or $\cos^2 x$)

$1 + \tan^2 x = \sec^2 x$

$\cot^2 x + 1 = \csc^2 x$

*Reciprocal*

$\sec x = \dfrac{1}{\cos x}$ *or* $\cos x \sec x = 1$

$\csc x = \dfrac{1}{\sin x}$ *or* $\sin x \csc x = 1$

*Odd-Even*

$\sin(-x) = -\sin x$ (odd)

$\cos(-x) = \cos x$ (even)

*Some more handy INTEGRALS:*

$\int \tan x\,dx = \ln|\sec x| + C$

$= -\ln|\cos x| + C$

$\int \sec x\,dx = \ln|\sec x + \tan x| + C$

### Taylor Series

If the function $f$ is "smooth" at $x = a$, then it can be approximated by the $n^{\text{th}}$ degree polynomial

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

### Lagrange Error Bound

If $P_n(x)$ is the $n^{\text{th}}$ degree Taylor polynomial of $f(x)$ about $c$ and $\left|f^{(n+1)}(t)\right| \le M$ for all $t$ between $x$ and $c$, then

$$\left|f(x) - P_n(x)\right| \le \frac{M}{(n+1)!}\left|x - c\right|^{n+1}$$

### Maclaurin Series

A Taylor Series about $x = 0$ is called Maclaurin.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

### Alternating Series Error Bound

If $S_N = \sum\limits_{k=1}^{N} (-1)^n a_n$ is the $N^{\text{th}}$ partial sum of a convergent alternating series, then

$$\left|S_\infty - S_N\right| \le \left|a_{N+1}\right|$$

### Geometric Series

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$$

diverges if $|r| \ge 1$; converges to $\dfrac{a}{1-r}$ if $|r| < 1$

This is available at http://covenantchristian.org/bird/Smart/Calc1/StuffMUSTknowColdNew.htm